Projection Operation for Multidimensional Geometric Modeling with Real Functions

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Abstract. We discuss different techniques to projecting geometric objects defined by real functions of several variables. The result of this operation is an object of the lower dimension with its own defining function. We discuss several approaches: analytical methods, approximate projections and global maximum searches. The accuracy is compared for two algorithms: the union of maximal cross-sections and the global search with the quadratic interpolation. The following applications are illustrated: 3D solid projection onto a 2D plane; 4D to 3D projection for sweeping by a moving solid; 3D reconstruction from the medial axis.

Introduction

This paper deals with the geometric projection operation in 3D modeling of surfaces/solids and in multidimensional geometric modeling. To project a surface or a solid in geometric modeling is a much more difficult problem than to project points and line segments in computer graphics. This operation has to result in a valid lower dimensional geometric model, which again can be used as an operand of further geometric transformations and analysis.

We consider the projection operation in the context of geometric modeling with real functions of several variables. The *function representation* (or *F-rep*) defines a geometric object by a single real continuous function of several variables as $F(X) \geq 0$. We do not require the defining function $F(X)$ to be polynomial or of any other specific type. The function can be defined with an analytical expression, or with a function evaluation algorithm, or with scattered data and an appropriate interpolation procedure. This representation combines many different models like classic "implicits", skeleton based "implicits", set-theoretic solids, sweeps, volumetric objects, parametric and procedural models [6],[9]. Set-theoretic operations are closed on this representation with the use of R-functions - C^k continuous definitions introduced by Rvachev [8] (see a survey in [10]). Many geometric operations are also closed on F-rep. They are blending, offsetting, Cartesian product, bijective mapping, metamorphosis and others (see [6] for details). These operations generate new real continuous defining functions and provide the closure property of the representation.

One of the main F-rep features is dimension independence. Geometric objects of various dimensions can be treated uniformly by most of operations.

There are also operations that change the object dimension: the Cartesian product increases and the projection decreases it. Several applications of the projection operation are well known:
• Traditional 3D to 2D projection;
• 4D to 3D projection for sweeping by a moving solid;
• Reconstruction from the medial axis transformation;
• Construction of an envelope of a parametric family;
• Implicitization of parametric curves and surfaces;
• Calculation of the surface area by integration.
Some more specific applications are known in path planning, optimization, catastrophe theory, systems of differential equations and statistical analysis.

Problem statement

To support real multidimensionality a geometric modeler has to provide the robust and effective projection operation. Usually the projection operation is defined by a linear transformation matrix A with $detA = 0$. This definition cannot be interpreted in terms of bijective mappings because it is not one-to-one mapping and an inverse mapping does not exist.

Let the initial object G_1 in E^n is defined by the real function $f_1(X_n) \geq 0$, where $X_n = (x_1, x_2, ..., x_i, ..., x_n)$. We call the geometric object G_2 a projection of G_1 to E^{n-1} with the defining function $f_2(X_{n-1})$, which is more or equal zero in the given point $X_{n-1} = (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_n)$ only if the point X_n exists where $f_1(X_n) \geq 0$. The object G_2 can be thought as a union of cross-sections of G_1 by the infinite set of hyperplanes $x_i = C$. The resulting function for the projection can also be defined as

$$f_2(X_{n-1}) = max_{x_i}(f_1(X_n)) \tag{1}$$

Let us formulate several requirements to the function $f_2(X_{n-1})$ and its evaluation algorithms:

- $f_2(X_{n-1})$ is a continuous real function;
- an evaluation algorithm should treat the initial function $f_1(X_n)$ as a "black box", i.e., a procedurally defined function with no preliminary information given about its nature or behavior;
- distance property of the function $f_2(X_{n-1})$ should be provided that is important for the further operations on the projection.

Other works

There are several different approaches to the projection operation in geometric modeling. The conversion of parametric curves and surfaces into implicit form can be considered as a projection operation. There are several analytical

conversion techniques such as resultants and Gröbner basis for the polynomial functions [2].

Kutsenko in [3] proposed the following defining function for the projection from 3D to 2D along z-axis:

$$\varphi^*(x,y) = \int_a^b F(x,y,z) + |F(x,y,z)|dz, \tag{2}$$

where $F(x,y,z)$ defines the initial 3D object and $\varphi^*(x,y)$ correctly defines the internal and boundary points of the projection. Although this approach is quite attractive for the analytical and numerical implementation, the behavior of the resulting function restricts its application in the F-rep framework. The function $\varphi^*(x,y)$ is equal zero everywhere outside the projection. Many operations such as blending, offsetting or metamorphosis require distance-like behavior of the defining function. So, the function $\varphi^*(x,y)$ is not enough satisfactory.

Analytical methods for eliminating a variable from two equations are also used in modeling sweeping by a moving solid. The analytical solutions were found in several specific cases of sweeping by a sphere and by a cylinder [13], [4]. Sweeping by a practically arbitrary solid is approximately modeled in [5]. The moving solid is converted to the approximate ray-representation, then the swept solid is represented as a union of series of generator instances (cross-sections). Note that here the projection operation is closed on the ray-representation.

Solving the problem of collision free path planning Wise and Bowyer [14] project an interference hypersolid into the configuration space. The result of this procedure is a recursive division of the configuration space (multi-dimensional generalization of bin-, quad- and octrees) into hyperboxes with different properties. So, the application of the projection operation requires here the conversion from the initial CSG model to the representation with a spatial occupancy enumeration.

Reconstruction of geometric objects from their medial axis is an actual problem in geometric modeling. The medial axis transformation represents an object as its lower dimensional skeleton with the radius of the maximal inscribable disk/sphere given at every point of the skeleton. The reconstruction of the original object from the medial axis can be thought as the projection operation from $n+k$-dimensional space into n-dimensional space, where k is a dimension of the skeleton and n is a dimension of the maximal disks. Papers [12] and [1] present the reconstruction algorithms resulting in the object boundary.

Proposed solutions

As it is mentioned in the previous section, the implementation of the projection operation often changes the representation of the object. Our goal here

is to propose several approaches that provide closure of this operation on the function representation.

Analytical approach: case study

Let us describe here the case study of the analytical reconstruction of the F-rep object from the medial axis transformation. A one-dimensional skeleton consisting of line segments is given in 3D space. For every point of the skeleton the radius of the maximal inscribable sphere is defined. The problem is to obtain a real function defining the original object. Let the skeleton consist of line segments with the variable sphere radius on it. Then, for a 4D object defined by j-th segment of the skeleton with endpoints (x_{1j}, y_{1j}, z_{1j}) and (x_{2j}, y_{2j}, z_{2j}) we get the description:

$$f_{1j}(x,y,z,t) = R_j^2(t) - (x - x_{0j}(t))^2 - (y - y_{0j}(t))^2 - (z - z_{0j}(t))^2, \quad (3)$$

where $x_{0j}(t) = x_{1j} + l_{xj}t$, $y_{0j}(t) = y_{1j} + l_{yj}t$, $z_{0j}(t) = z_{1j} + l_{zj}t$, and $l_{xj} = x_{2j} - x_{1j}$, $l_{yj} = y_{2j} - y_{1j}$, $l_{zj} = z_{2j} - z_{1j}$. To project this object into 3D space following Equation (1), we have to find the maximal value of the function $f_{1j}(x,y,z,t)$ for the given 3D point (x,y,z). Assuming the linear function $R_j(t) = R_{1j}(1-t) + R_{2j}t$, it can be shown that the following parameter value corresponds to the function f_{1j} maximum:

$$t_{0j} = \frac{l_{xj}(x - x_{1j}) + l_{yj}(y - y_{1j}) + l_{zj}(z - z_{1j}) - R_{1j}(R_{2j} - R_{1j})}{l_{xj}^2 + l_{yj}^2 + l_{zj}^2 + (R_{2j} - R_{1j})^2} \quad (4)$$

Then, for the whole reconstructed 3D object the defining function is

$$f_2(x,y,z) = max_j(f_{1j}(x,y,z,t_{0j})). \quad (5)$$

Figure 1 shows a 3D object reconstructed from its skeleton using this approach. The skeleton was modeled with "random walk" calculations with transition probabilities depending on the direction of motion. This type of skeleton can be observed in plants, blood vessels and porous media.

Numerical algorithms

We discuss and compare here the numerical evaluation algorithms for the defining function of the projection. They are approximation and global maximum search algorithms. These algorithms deal with the given procedurally defined initial function $f_1(X_n)$ and evaluate the function $f_2(X_{n-1})$ of the projection in the given point.

- **Union of cross-sections**. By selecting a finite set of hyperplanes $x_i = C_j$ one can approximate the projection result applying R-union functions:

$$f_2(X_{n-1}) = (f_{11} \vee f_{12}) \ldots \vee f_{1j}) \ldots \vee f_{1N}), \quad (6)$$

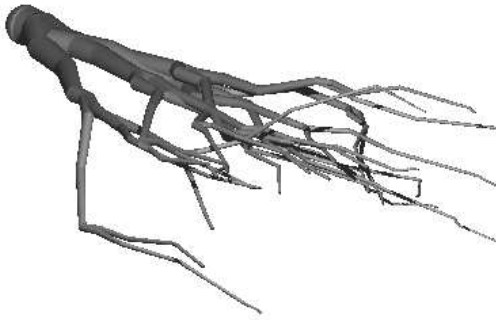


Fig. 1. Analytical 3D reconstruction from a skeleton

where $f_{1j} = f_1(x_1, x_2, \ldots, x_{i-1}, C_j, x_{i-1}, \ldots, x_n)$, N is the number of cross-sections and

$$h \vee g = h + g + \sqrt{h^2 + g^2} \tag{7}$$

is one of the R-functions proposed by Rvachev [8] for the exact description of the set-theoretic union of two objects described by real functions h and g. The accuracy of the result of this projection operation highly depends on the step between adjacent cross-sections. Figure 2 shows a 3D to 2D projection of a tubular surface, which is defined as a 2D disk changing its position in xy-plane and moving in z-direction. Figure 2 (a-c) illustrates the comparison of three different algorithms (see the discussion below).

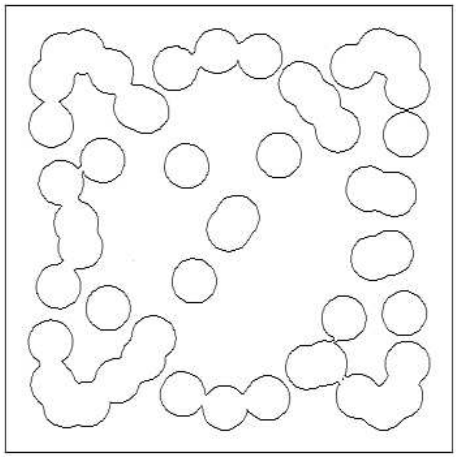
- **Blending union**. One can try to change the standard union to the blending union operation [6]:

$$h \vee_b g = h + g + \sqrt{h^2 + g^2} + \frac{a_0}{1 + (\frac{h}{a_1})^2 + (\frac{g}{a_2})^2}. \tag{8}$$

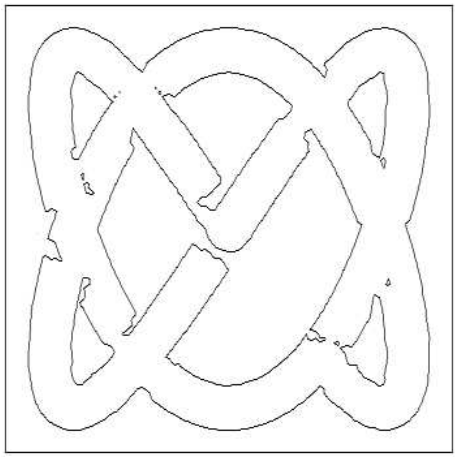
Blending union of cross-sections can produce much better result than standard union but the choice of parameters a_0, a_1, a_2 depends very much on geometry of the projected object (for example, on variations in the skeleton curvature). There is no any automatic method yet to choose these parameters and to establish their dependance on the features of the initial object.

- **Union of maximal cross-sections**. This approximate projection applies set-theoretic union to the interpolation terms between adjacent cross-sections taken with a regular step:

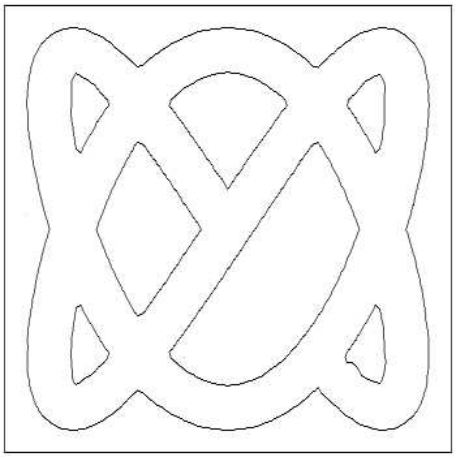
$$f_2(X_{n-1}) = (f_{11} \vee f^*_{11}) \vee f^*_{12}) \ldots \vee f^*_{1j}) \ldots \vee f^*_{1,N-2}) \vee f_{1,N}, \tag{9}$$



(a) Union of cross-sections



(b) Global maximum search with the quadratic interpolation



(c) Union of maximal cross-sections

Fig. 2. 3D to 2D projection of a tubular surface by three algorithms using 40 cross-sections

where N is number of cross-sections,

$$f_{1j}^* = f_1(x_1, x_2, \ldots, x_{i-1}, C_j^*, x_{i-1}, \ldots, x_n). \tag{10}$$

Here the constant $C_j^* = C_j + C_0 dx_i$ defines the maximal function $f_1(X_n)$ between three cross-sections, where C_j is the value of x_i in the j-th grid node with the grid step dx_i. Parameter C_0 is calculated using the quadratic interpolation:

$$C_0 = \frac{1}{2} - \frac{f_{1,j+1} - f_{1,j}}{f_{1,j+2} - 2f_{1,j+1} + f_1 j} \tag{11}$$

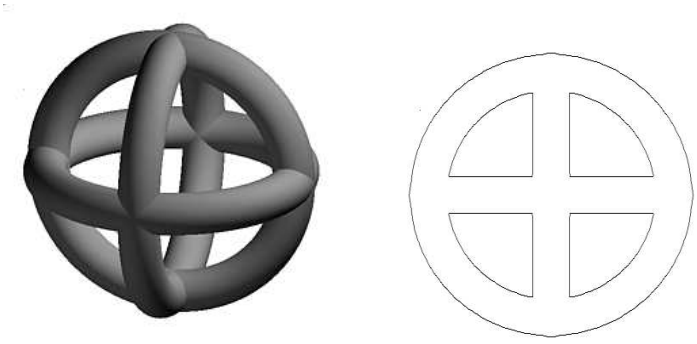
Note that if $C_0 < 0$, then $f_{1j}^* = f_{1j}$, and if $C_0 > 2$, then $f_{1j}^* = f_{1,j+2}$. Figure 2c shows a quite good result obtained with this algorithm.

- **Global maximum search**. The projection problem can be formulated as a global extremum search (see Equation (1)). We have tested an algorithm that applies the Newton quadratic interpolation with Equations (10,11) only once at the point of maximal $f_1(X_n)$ taken from the regular grid of samples [11]. This algorithm obviously requires fewer number of function $f_1(X_n)$ evaluations than the previous one but has shown more poor quality of the result for our test operations (see Figure 2b and the accuracy comparison below).
- **Combination of random and quadratic global searches**. It is well known that a combination of random and gradient/sequential searches is an effective approach to the global maximum search. We apply quasi-random Sobol sequences (see [7] for the description and the C-code) to generate an initial estimation C_j for the subsequent quadratic search with Equations (10,11). The search stops when the difference between two found maxima does not exceed the given threshold. This algorithm works more fast (about 20%) than the previous one for the test projection (Figure 2) but its behavior highly depends on the threshold selection.

We have selected two algorithms to compare their accuracy: the union of maximal cross-sections and the global search with the quadratic interpolation. The test problem is shown in Figure 3: a R-union of three tori is projected into a plane orthogonal to an axis of one of tori. We have selected 16 points in the projection where the resulting function has to be zero. Figure 4 shows the plots of the integral error in these points versus the number of cross-sections for both methods. Evidently, the union of maximal cross-sections is much more stable and converges very fast. Figure 5 illustrates a 4D to 3D projection with this method: a solid swept by a rotating cube which moves along the triangle trajectory.

Conclusion

Projection operation is important for constructing a multidimensional geometric modeler. It is also quite useful in several applications. This operation



(a) Union of three tori

(b) Projection into xy-plane

Fig. 3. Projection of three tori into a 2D plane

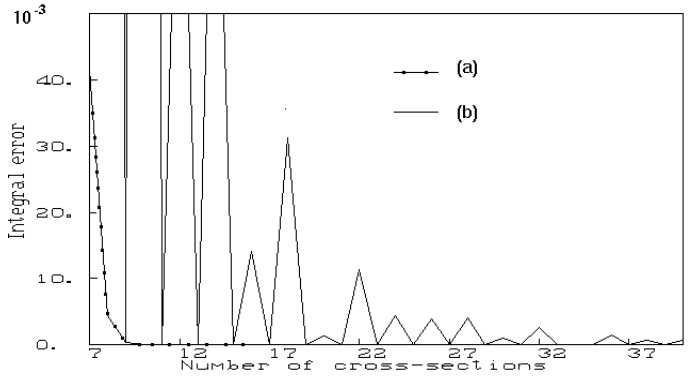


Fig. 4. Integral errors for two projection algorithms: (a) union of maximal cross-sections; (b) global search with the quadratic interpolation

can be generally expressed in terms of real functions. Several numerical algorithms exist for the evaluation of real functions defining projections. These algorithms are time consuming because they multiply evaluate the object's defining function. This can significantly slow down the following operations on projections. To effectively utilize the general approach proposed in this paper one can incorporate it into a modeler with a voxel- or other approximate model generation for the resulting projection and with an appropriate

Fig. 5. 4D to 3D projection: sweep by a moving cube

interpolation procedure to ensure the closure property of F-rep. The central problem for our future research on the projection operation is selecting a more effective and stable global search algorithm and exploring different searches on parallel computers.

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